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GENERALIZED NOETHER'S FORMULAS FOR PLANE CURVES SINGULARITIES

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Abstract. Let C be a plane analytic curve defined in a neighborhood of the origin and let $\pi : M \to \mathbb{C}^2$ be a local toric modification. We give a formula which connects the number $\delta_0(C)$ of double points with the sum $\sum_p \delta_p(\tilde{C})$, which runs over the intersection points of the proper preimage $\tilde{C}$ of C with the exceptional divisor $\pi^{-1}(0)$. We also obtain similar formulas for the Milnor number $\mu_0(C)$ and the intersection multiplicity $(C, D)_0$ of two plane analytic curves. Our results generalize the Kouchnirenko and Bernstein theorems and Noether's classical formulas for the delta invariant and the intersection multiplicity.

1. Kouchnirenko and Bernstein theorems. Let $f \in \mathbb{C}\{x, y\}$ be a convergent power series such that $f(0,0) = 0$ and $f(x, 0)$, $f(0, y)$ do not vanish. Such a series is called convenient. If $f(x, y) = \sum_{ij} a_{ij} x^i y^j$ is a convenient power series, then by definition the Newton diagram Δ_f is the convex hull of the set $\bigcup_{a_{ij} \neq 0} \{\, (i, j) + \mathbb{R}^2_+ \,\}$. The union of compact edges of Δ_f is called the Newton polygon of f and is denoted $\mathcal{N}_f$.

Let us introduce further notations. For the Newton diagram Δ whose Newton polygon $\mathcal{N}$ touches axes at points $(a, 0)$ and $(0, b)$, denote:

- $A(\Delta) = \mathrm{Area}\,(\mathbb{R}^2_+ \setminus \Delta)$,
- $\mu(\Delta) = 2A(\Delta) - a - b + 1$,
- $r(\Delta) =$ number of lattice points on $\mathcal{N}$ minus 1,
- $\delta(\Delta) = (\mu(\Delta) + r(\Delta) - 1)/2$.

If Δ_1, Δ_2 are two Newton diagrams, then their mixed Minkowski area is the quantity $[\Delta_1, \Delta_2] = A(\Delta_1 + \Delta_2) - A(\Delta_1) - A(\Delta_2)$.

Remark 1. Let Δ' be the closure of the set $\mathbb{R}^2_+ \setminus \Delta$. Then by Pick's theorem, $A(\Delta) = I(\Delta') + B(\Delta')/2 - 1$, where $I(\Delta')$ is the number of interior lattice points of the polygon Δ' and $B(\Delta')$ is the number of lattice points on the border of Δ'. Let $N(\Delta')$ be the number of lattice points on the Newton polygon $\mathcal{N}$ of Δ. Using Pick's formula, it is easy to check that $\mu(\Delta) = 2I(\Delta') + N(\Delta') - 2$ and $\delta(\Delta) = I(\Delta') + N(\Delta') - 2$.

In the 1970s, mathematicians from Arnold's seminar gave many formulas for invariants of singularities in terms of Newton diagrams. We quote some of their results (see [**2, 7, 6**]).

THEOREM 1. *Let $f, g \in \mathbb{C}\{x, y\}$ be convenient convergent power series. Then*

$$(f, g)_0 \geq [\Delta_f, \Delta_g], \tag{1}$$

$$\delta_0(f) \geq \delta(\Delta_f), \tag{2}$$

$$\mu_0(f) \geq \mu(\Delta_f), \tag{3}$$

$$r_0(f) \leq r(\Delta_f). \tag{4}$$

If coefficients of $f(x, y)$, $g(x, y)$, with indices from the sets $\mathcal{N}_f$, $\mathcal{N}_g$, respectively, are sufficiently general, then (1)–(4) become equalities.

In Theorem 1: $(f, g)_0$ is the intersection multiplicity of f and g, $\delta_0(f)$ is the number of double points of a curve $f = 0$, $\mu_0(f)$ is the Milnor number of f (see [**3**], p. 573 for definitions of $\delta_0(f)$ and $\mu_0(f)$) and $r_0(f)$ is the number of branches of $f = 0$.

The power series f, g are "sufficiently general" if their coefficients of indices from the sets $\mathcal{N}_f$, $\mathcal{N}_g$, respectively, satisfy some algebraic inequalities. These inequalities are called nondegeneracy conditions (see [**6**]).

EXAMPLE 1. Consider power series $f(x, y) = x^m + y^m$ and $g(x, y) = x^n + 2y^n$. The Newton polygons of Δ_f and Δ_q are segments with endpoints $(m, 0)$, $(0, m)$ (for $\mathcal{N}_f$) and $(n, 0)$, $(0, n)$ (for $\mathcal{N}_g$). There is $\delta(\Delta_f) = m(m-1)/2$, $\mu(\Delta_f) = (m-1)^2$, $r(\Delta_f) = m$ and $[\Delta_f, \Delta_g] = mn$. One can check that f and g satisfy nondegeneracy conditions. Hence, by Theorem 1 their invariants of singularities are equal to the corresponding quantities for their Newton diagrams.

2. Noether's formulas. Let C, D be analytic curves defined in a neighborhood of the origin of the complex plane. Let $\sigma : M \to \mathbb{C}^2$ be a blowing-up of $\mathbb{C}^2$ at 0. Noether's classical formulas (see [**4**], Theorem 3.11.12 and Lemma 3.3.4) relate invariants of singularities of these curves to invariants of singularities of their proper preimages.

Theorem 2. *Let $\tilde{C}$, $\tilde{D}$ be proper preimages of curves C and D under blowing-up σ. Then*

$$\delta_0(C) = m_0(C)(m_0(C)-1)/2 + \sum_{p\in\sigma^{-1}(0)} \delta_p(\tilde{C}), \tag{5}$$

$$(C,D)_0 = m_0(C)m_0(D) + \sum_{p\in\sigma^{-1}(0)} (\tilde{C},\tilde{D})_p. \tag{6}$$

In the above formulas, $\delta_p(\tilde{C})$ and $(\tilde{C},\tilde{D})_p$ denote the delta invariant and the intersection multiplicity at p and $m_0(C)$ is the multiplicity of C at 0 (i.e. the order of the defining equation of this curve). We also apply standard conventions on the symbol $+\infty$, i.e., the left-hand side of one of the above equations is $+\infty$ if and only if one of the summands on the right hand side is $+\infty$.

3. Local toric modifications. In this section we introduce local toric modifications of $\mathbb{C}^2$ (see [**6, 8**]). We also define thickened Newton diagrams in order to generalize Theorem 2 onto an arbitrary local toric modification $\pi : M \to \mathbb{C}^2$.

3.1. *Fans.* By a simple cone $\sigma \subset \mathbb{R}^2$ we mean the set $\sigma = \sigma[\vec{\xi},\vec{\nu}] = \{\, \alpha\vec{\xi} + \beta\vec{\nu} : \alpha \geq 0, \beta \geq 0 \,\}$, where vectors $\vec{\xi}$, $\vec{\nu} \in \mathbb{R}^2$ have integer coordinates and $\det(\vec{\xi},\vec{\nu}) = \pm 1$.

By a fan we mean a finite set of simple cones such that their union is the first quadrant $\mathbb{R}^2_+$ and such that they intersect at most along edges. For every fan $\mathcal{W}$ consisting of n cones, one can enumerate counter clockwise the shortest lattice vectors in their rays. We get a sequence $\vec{\xi_0}$, $\vec{\xi_1}$, …, $\vec{\xi_n}$, where $\vec{\xi_0} = (1,0)$, $\vec{\xi_n} = (0,1)$ and $\det(\vec{\xi}_{i-1},\vec{\xi_i}) = 1$ for $1 \leq i \leq n$. We say that $\mathcal{W}$ is spanned by $\vec{\xi_0}$, $\vec{\xi_1}$, …, $\vec{\xi_n}$.

3.2. *Local toric modifications.* With every simple cone $\sigma = \sigma[\vec{\xi},\vec{\nu}]$, where $\vec{\xi} = (\xi_1,\xi_2)$, $\vec{\nu} = (\nu_1,\nu_2)$, $\det(\vec{\xi},\vec{\nu}) = 1$, we associate a mapping $\varphi_\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ given in coordinates by

$$\varphi_\sigma : \begin{cases} x = u^{\xi_1}v^{\nu_1}, \\ y = u^{\xi_2}v^{\nu_2}. \end{cases}$$

Let $\mathcal{W}$ be a fan consisting of n cones.

Theorem 3. *There exist a smooth analytic manifold M and a proper analytic mapping $\pi : M \to \mathbb{C}^2$ such that:*

(i) π is an isomorphism from $M \setminus \pi^{-1}(0)$ to $\mathbb{C}^2 \setminus \{0\}$,

(ii) the manifold M is covered by n charts associated with cones σ_i ($i = 1 \dots n$) and in local coordinates of the i-th chart, $\pi(u,v) = \varphi_{\sigma_i}(u,v)$.

We call $\pi : M \to \mathbb{C}^2$ a local toric modification associated with a fan $\mathcal{W}$.

EXAMPLE 2. (see [**1**], page 178). Consider a fan $\mathcal{W}$ spanned by $\vec{\xi_0} = (1,0)$, $\vec{\xi_1} = (1,1)$ and $\vec{\xi_2} = (0,1)$. Then the toric modification $\pi : M \to \mathbb{C}^2$ associated with $\mathcal{W}$ is a blowing-up of $\mathbb{C}^2$ at zero. Indeed the manifold M is covered by two charts associated with two cones of $\mathcal{W}$ and the mapping π is given by a formula $(x,y) = (uv, v)$ in the first chart and by a formula $(x,y) = (u, uv)$ in the second.

3.3. *Thickened Newton diagrams.* For a Newton diagram Δ and $\vec{\xi} \in \mathbb{R}^2_+$, we define the support function

$$l(\Delta, \vec{\xi}) = \inf_{p \in \Delta} \langle p, \vec{\xi} \rangle.$$

For a fan $\mathcal{W}$ spanned by vectors $\vec{\xi_0}$, $\vec{\xi_1}$, ..., $\vec{\xi_n}$ and a Newton diagram Δ, we define $\tilde{\Delta}$ as the intersection of $n+1$ half-planes

$$\tilde{\Delta} = \bigcap_{i=0}^{n} \{\, p \in \mathbb{R}^2 : \langle p, \vec{\xi_i} \rangle \geq l(\Delta, \vec{\xi_i}) \,\}$$

and call this set the *thickened Newton diagram relative to* $\mathcal{W}$. It follows directly from the definition of $\tilde{\Delta}$ that $l(\tilde{\Delta}, \vec{\xi_i}) = l(\Delta, \vec{\xi_i})$ for $i = 0, \ldots, n$.

4. Generalized Noether's formulas.

THEOREM 4. *Let $\pi : M \to \mathbb{C}^2$ be a local toric modification associated with a fan $\mathcal{W}$. Let f, $g \in \mathbb{C}\{x,y\}$ be convenient power series. Take plane analytic curves C, D given by $f(x,y) = 0$, $g(x,y) = 0$, respectively, and their proper preimages $\tilde{C}$, $\tilde{D}$ under π. Then*

$$\delta_0(C) = \delta(\tilde{\Delta}_f) + \sum_{p \in \pi^{-1}(0)} \delta_p(\tilde{C}), \tag{7}$$

$$(C, D)_0 = [\tilde{\Delta}_f, \tilde{\Delta}_g] + \sum_{p \in \pi^{-1}(0)} (\tilde{C}, \tilde{D})_p. \tag{8}$$

REMARK 2. Keep the notations and assumptions of Theorem 4 and assume that $\pi : M \to \mathbb{C}^2$ is a blowing-up (see Example 2). For every convenient power series $h \in \mathbb{C}\{x,y\}$ the Newton polygon of a thickened Newton diagram $\tilde{\Delta}_h$ is a segment with endpoints $(\operatorname{ord} h, 0)$ and $(0, \operatorname{ord} h)$. Hence, $\delta(\tilde{\Delta}_f) = m_0(C)(m_0(C) - 1)/2$ and $[\tilde{\Delta}_f, \tilde{\Delta}_g] = m_0(C)m_0(D)$. We see that for the blowing-up, formulas (7) and (8) from Theorem 4 reduce to Noether's formulas (5) and (6).

REMARK 3. Let $\pi : M \to \mathbb{C}^2$ be a local toric modification associated with a fan $\mathcal{W}$ and let f, g be convenient power series in two complex variables such that $\tilde{\Delta}_f = \Delta_f$ and $\tilde{\Delta}_g = \Delta_g$. Assume that f and g satisfy nondegeneracy conditions. Then by [**6**], the curves $\tilde{C}$ and $\tilde{D}$ are smooth and do not have joint points on exceptional divisor $\pi^{-1}(0)$. Thus, in this particular case, Theorem 4 follows from (2) and (1) of Theorem 1, because the sums on the right-hand side of (7) and (8) are 0.

Corollary 5. *Under the assumptions and notations of Theorem 4*

$$\mu_0(C) = \mu(\tilde{\Delta}_f) + r(\tilde{\Delta}_f) + \sum_{p \in \tilde{C} \cap \pi^{-1}(0)} (\mu_p(\tilde{C}) - 1). \tag{9}$$

4.1. *A decomposition of a toric modification to blowing-ups.* In this subsection, we check that every local toric modification can be decomposed into a finite number of blowing-ups (see for example [**9**], Proposition (2.8)). We will apply this (well-known) fact in an inductive proof of Theorem 4.

Let us start with a lemma describing mutual positions of vectors spanning a fan.

Lemma 6. *If a fan $\mathcal{W}$ is spanned by vectors $\vec{\xi}_0, \vec{\xi}_1, \ldots, \vec{\xi}_n$ and the vectors $\vec{\xi}_k, \vec{\xi}_l$ $(k+1 < l)$ form a basis of a lattice, then one of the vectors $\vec{\xi}_i$ is equal to $\vec{\xi}_k + \vec{\xi}_l$.*

Proof. Suppose that this is not the case. Then $\vec{\xi}_k + \vec{\xi}_l$ is inside one of the cones σ_j of $\mathcal{W}$, where $k < j \le l$. Therefore, we have the following equations with non-negative integer coefficients:

$$\begin{array}{rcl} \vec{\xi}_{j-1} & = & n_1\vec{\xi}_k + n_2\vec{\xi}_l, \\ \vec{\xi}_j & = & m_1\vec{\xi}_k + m_2\vec{\xi}_l, \\ \vec{\xi}_k + \vec{\xi}_l & = & a\vec{\xi}_{j-1} + b\vec{\xi}_j, \end{array}$$

where $a > 0$, $b > 0$. Substituting $\vec{\xi}_{j-1}$, $\vec{\xi}_j$ from the first two equations to the third one, we get $an_1 + bm_1 = 1$ and $an_2 + bm_2 = 1$.

Assume that $j < l$. Then $m_1 > 0$, $m_2 > 0$ (because $\vec{\xi}_j$ is inside the cone $\sigma[\vec{\xi}_k, \vec{\xi}_l]$) and by above equalities we get $n_1 = n_2 = 0$. Hence, $\vec{\xi}_{j-1} = \vec{0}$ and we arrive at a contradiction.

If $j = l$, then $\vec{\xi}_{j-1}$ is inside the cone $\sigma[\vec{\xi}_k, \vec{\xi}_l]$ and similar computation gives $\vec{\xi}_j = \vec{0}$. Contradiction. □

Corollary 7. *If a fan $\mathcal{W}_n$ is spanned by vectors $\vec{\xi}_0, \vec{\xi}_1, \ldots, \vec{\xi}_n$ then for some i $(1 \le i < n)$ there is*

$$\vec{\xi}_i = \vec{\xi}_{i-1} + \vec{\xi}_{i+1}.$$

The proof of a corollary uses a simple recurrence. The vectors $\vec{\xi}_0$ and $\vec{\xi}_n$ satisfy assumptions of Lemma 6. Hence, among the vectors spanning the fan there is one of the form $\vec{\xi}_j = \vec{\xi}_0 + \vec{\xi}_n$. Next we apply Lemma 6 to a pair $\vec{\xi}_0$, $\vec{\xi}_j$

or to a pair $\vec{\xi}_j$, $\vec{\xi}_n$. Continuing this procedure, we arrive at such vectors $\vec{\xi}_{i-1}$, $\vec{\xi}_i$, $\vec{\xi}_{i+1}$ that $\vec{\xi}_i = \vec{\xi}_{i-1} + \vec{\xi}_{i+1}$.

It follows from Corollary 7 that for every fan $\mathcal{W}_{n+1}$ consisting of $n+1$ cones there exist a fan $\mathcal{W}_n$ consisting of n cones such that all cones of $\mathcal{W}_n$ but one belong to $\mathcal{W}_{n+1}$ and one of the cones $\sigma = \sigma[\vec{\xi}, \vec{\nu}] \in \mathcal{W}_n$ decomposes into two cones $\sigma[\vec{\xi}, \vec{\xi} + \vec{\nu}]$, $\sigma[\vec{\xi} + \vec{\nu}, \vec{\nu}] \in \mathcal{W}_{n+1}$. We call $\mathcal{W}_{n+1}$ a subdivision of $\mathcal{W}_n$.

THEOREM 8. *Let $\pi_n : M_n \to \mathbb{C}^2$, $\pi_{n+1} : M_{n+1} \to \mathbb{C}^2$ be toric modifications associated with a fan $\mathcal{W}_n$ and its subdivision $\mathcal{W}_{n+1}$. Let $\tilde{\sigma}_i$ be the cone of $\mathcal{W}_n$, which is divided into two. Then $\pi_{n+1} = \pi_n \circ \sigma$, where σ is a blowing-up of M_n at the origin of the local coordinate system associated with $\tilde{\sigma}$.*

PROOF. Let $\tilde{\sigma} = \sigma[\vec{\xi}, \vec{\nu}]$, $\sigma' = \sigma[\vec{\xi}, \vec{\xi} + \vec{\nu}]$, $\sigma'' = \sigma[\vec{\xi} + \vec{\nu}, \vec{\nu}]$. In all charts associated with cones different from $\tilde{\sigma}$, σ', σ'', mappings π_n and π_{n+1} are given by identical formulas, hence, in these charts σ is an identity. It is easy to check that $\varphi_{\sigma'} = \varphi_{\tilde{\sigma}} \circ \sigma$, where $\sigma(u, v) = (uv, v)$, and $\varphi_{\sigma''} = \varphi_{\tilde{\sigma}} \circ \sigma$, where $\sigma(u, v) = (u, uv)$. Hence, σ is a blowing-up of M_n at a point $(0, 0)$ of a chart associated with $\tilde{\sigma}$. $\square$

4.2. *Orders of proper preimages in centers of blowing-ups.*

LEMMA 9. *Let $\pi : M \to \mathbb{C}^2$ be a local toric modification associated with a fan $\mathcal{W}$ and let $\sigma = \sigma[\vec{\xi}, \vec{\nu}]$ be a cone of $\mathcal{W}$. Let $q \in M$ be the origin of the chart associated with σ. Let $f \in \mathbb{C}\{x, y\}$ be a convenient power series with a Newton diagram Δ. Let $\tilde{C}$ be the proper preimage of the plane analytic curve C given by $f(x, y) = 0$. Then $m_q(\tilde{C}) = l(\Delta, \vec{\xi} + \vec{\nu}) - l(\Delta, \vec{\xi}) - l(\Delta, \vec{\nu})$.*

PROOF. If $f(x,y) = \sum_{ij} a_{ij} x^i y^j$ then $(f \circ \varphi_\sigma)(u, v) = \sum_{ij} a_{ij} u^{\langle (i,j), \vec{\xi} \rangle} v^{\langle (i,j), \vec{\nu} \rangle}$. Excluding the highest possible powers of variables u, v from the sum above, we get

$$(f \circ \varphi_\sigma)(u, v) = u^{l(\Delta, \vec{\xi})} v^{l(\Delta, \vec{\nu})} \tilde{f}(u, v), \tag{10}$$

where $\tilde{f}(u, v) = 0$ is an equation for $\tilde{C}$ in a chart associated with σ.

Take an arc $\gamma : (\mathbb{C}, 0) \to (M, p)$ given by $(u, v) = (c_1 t, c_2 t)$. Then for generic constants c_1, c_2 there is $m_q(\tilde{C}) = \operatorname{ord} \tilde{f} = \operatorname{ord} \tilde{f} \circ \gamma$. One easily checks that $\varphi_\sigma \circ \gamma$ is given by $(x, y) = (d_1 t^{\xi_1 + \nu_1}, d_2 t^{\xi_2 + \nu_2})$, where $d_1 = c_1^{\xi_1} c_2^{\nu_1}$ and $d_2 = c_1^{\xi_2} c_2^{\nu_2}$. For generic d_1, d_2 we have $\operatorname{ord}_t f(d_1 t^{\xi_1 + \nu_1}, d_2 t^{\xi_2 + \nu_2}) = l(\Delta, \vec{\xi} + \vec{\nu})$ (see [**6**]). Thus, substituting γ to (10) and computing the orders, we get

$$l(\Delta, \vec{\xi} + \vec{\nu}) = \operatorname{ord} \left(t^{l(\Delta, \vec{\xi})} t^{l(\Delta, \vec{\nu})} \tilde{f}(\gamma(t)) \right) = l(\Delta, \vec{\xi}) + l(\Delta, \vec{\nu}) + \operatorname{ord} \tilde{f},$$

which proves the Lemma. $\square$

4.3. *Proof of Theorem 4.*

Proof of (7). Let $\mathcal{W}_n$, $\mathcal{W}_{n+1}$ be fans from Theorem 8. Let f_1, f_2 be convenient power series such that $\tilde{\Delta}_{f_i} = \tilde{\Delta}$, where thickened Newton diagrams are relative to $\mathcal{W}_{n+1}$ for $i = 1, 2$. Let C_i be plane curves given by $f_i(x, y) = 0$ for $i = 1, 2$. Keeping notation from Theorem 8, we shall check that the formula

$$(11) \qquad \delta_0(C_1) - \sum_{p \in \pi_n^{-1}(0)} \delta_p(\tilde{C}_1) \quad = \quad \delta_0(C_2) - \sum_{p \in \pi_n^{-1}(0)} \delta_p(\tilde{C}_2)$$

implies

$$(12) \qquad \delta_0(C_1) - \sum_{p \in \pi_{n+1}^{-1}(0)} \delta_p(\hat{C}_1) \quad = \quad \delta_0(C_2) - \sum_{p \in \pi_{n+1}^{-1}(0)} \delta_p(\hat{C}_2).$$

Here, $\tilde{C}_i$ and $\hat{C}_i$ are proper preimages of the curves C_i on manifolds M_n and M_{n+1}, respectively.

By Theorem 8, $\pi_{n+1} = \pi_n \circ \sigma$, where σ is a blowing-up of M_n at the center q, which is the origin of the chart associated with $\tilde{\sigma} = \sigma[\vec{\xi}, \vec{\nu}]$. By Lemma 9, $m_q(\tilde{C}_i) = l(\tilde{\Delta}, \vec{\xi} + \vec{\nu}) - l(\tilde{\Delta}, \vec{\xi}) - l(\tilde{\Delta}, \vec{\nu})$ for $i = 1, 2$. Therefore, from Noether's formula (5) we get

$$(13) \qquad \delta_q(\tilde{C}_1) - \sum_{p \in \sigma^{-1}(q)} \delta_p(\hat{C}_1) \quad = \quad \delta_q(\tilde{C}_2) - \sum_{p \in \sigma^{-1}(q)} \delta_p(\hat{C}_2)$$

Adding (11) and (13), we get (12).

An inductive argument with respect to the number of cones in the fan leads to the conclusion that formula (11) holds for every fan $\mathcal{W}$ and all convenient power series f_1, f_2 such that $\tilde{\Delta}_{f_1} = \tilde{\Delta}_{f_2}$. Taking as f_2 a nondegenerate convergent power series with a Newton diagram $\tilde{\Delta}$ and using Theorem 1, we get $\delta_0(C_2) = \delta(\tilde{\Delta})$ and $\delta_p(\tilde{C}_2) = 0$ for every $p \in \pi_n^{-1}(0)$ (see also Remark 3). Hence, the right-hand side of (11) is equal to $\delta(\tilde{\Delta})$, which proves (7). □

Proof of (8). We only outline the proof briefly, because it is analogous to the proof of (7). First, we show by induction on the number of cones that

$$(14) \qquad (C_1, D_1)_0 - \sum_{p \in \pi^{-1}(0)} (\tilde{C}_1, \tilde{D}_1)_p \quad = \quad (C_2, D_2)_0 - \sum_{p \in \pi^{-1}(0)} (\tilde{C}_2, \tilde{D}_2)_p,$$

where the curves C_i, D_i are given by $f_i(x, y) = 0$, $g_i(x, y) = 0$ ($i = 1, 2$) and $\tilde{\Delta}_{f_1} = \tilde{\Delta}_{f_2}$, $\tilde{\Delta}_{g_1} = \tilde{\Delta}_{g_2}$. In the inductive proof, we use formula (6) for the intersection multiplicity.

Then we take as f_2 and g_2 nondegenerate power series with Newton diagrams $\Delta_{f_2} = \tilde{\Delta}_{f_1}$ and $\Delta_{g_2} = \tilde{\Delta}_{g_1}$. By Theorem 1 and Remark 3, the right-hand side of (14) is equal to $[\Delta_{f_2}, \Delta_{g_2}]$, which proves (8). □

4.4. *Proof of Corollary 5.* It is enough to use a well-known formula $\delta_p(C) = (\mu_p(C) + r_p(C) - 1)/2$. After substituting it to (7) and multiplying by 2, we get

$$\mu_0(C) + r_0(C) - 1 \quad = \quad 2\delta(\tilde{\Delta}_f) + \sum_{p \in \tilde{C} \cap \pi^{-1}(0)} (\mu_p(\tilde{C}) + r_p(\tilde{C}) - 1).$$

Then using the formula $r_0(C) = \sum_{p \in \pi^{-1}(0)} r_p(\tilde{C})$ and the definition of $\delta(\tilde{\Delta}_f)$, we get (9). □

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